Closed Form, Exact Solutions of the Schrödinger Equation with an $|x|$ Potential

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Abstract

We derive an exact solution of the Schrödinger Equation when the potential is given by $V(x) = g|x|$, with g a constant controlling the units and scaling of the potential. The solution is exact in terms of transcendental functions, specifically the Airy function and its derivatives. This solution is novel because the energy eigenvalues are determined by continuity requirements on the wave function and its derivative at $x = 0$, which also control the singular behavior arising from the form of the potential at the origin. We also show a calculation of the energy eigenvalues by a direct operator-based method. We explore the properties of this solution in comparison with the harmonic oscillator. This exact solution was encountered in research on the solution of the Schrödinger-Poisson system in solid-state physics and in quantum gravity and we believe it may have utility in other problem areas as well.

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I. INTRODUCTION

Beyond their intrinsic mathematical interest, exact solutions of the Schrödinger Equation provide useful building blocks in constructing models in quantum mechanics. Numerical solutions account for the overwhelming majority of work in quantum mechanical theory owing to the difficulty of producing solutions for all but a handful of simple cases: the Coulomb potential, the simple harmonic oscillator, particle-in-a-box, and a very few others. A closed form, analytic solution affords the luxury, in appropriate circumstances, of a reference case calculated by completely separate means, which is a useful conceptual and numerical standard of comparison.

This solution was identified in connection with work on the Schrödinger-Poisson system, which is relevant to work in solid-state physics [1–3] and quantum gravity [4–7]. The Schrödinger-Poisson system is nonlinear and poses some special problems as a result. The solution we develop in this paper is a stationary solution for the Schrödinger Equation alone, with a specified potential, and so nonlinearity is not a problem, though the solution involves a somewhat complicated set of expressions in transcendental functions.

II. DEVELOPMENT OF SOLUTION

A. The Wave Function

We begin with the time dependent Schrödinger equation in one spatial dimension,

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} + mg|x|\psi, \tag{1}$$

where m is the mass of the particle. We have chosen units $\hbar = 1$ and introduced the potential term, $V(x) = mg|x|$, with g a constant controlling the scale of the potential. We have written this problem in the form of motion around a one-dimensional gravitational point source, but it may also be thought of in terms of an equivalent problem with electric fields. Since we are interested in stationary states, we apply a time dependence of the form,

$$\psi(x,t) = A(x)e^{-iEt}, \tag{2}$$

with E the energy of the stationary state. With this form of the time dependence, the stationary Schrödinger Equation for the amplitude becomes

$$EA = -\frac{1}{2m}\frac{d^2A}{dx^2} + mg|x|A. \tag{3}$$

Without loss of generality, we can take $A(x)$ to be real.

Equation (3) has a solution in terms of the Airy functions, $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ [8, 9], with an argument of the Airy functions of $(2m^2g)^{1/3}|x| - \left(\frac{2}{mg^2}\right)^{1/3} E$. Finiteness of the wave function as $x \to \pm\infty$ implies that the integration constant multiplying the $\mathrm{Bi}(z)$ term is zero, while the value of c, the integration constant multiplying the $\mathrm{Ai}(z)$ term, may be determined from the wave function normalization condition, $\int_{-\infty}^{\infty} [A(x)]^2\, dx = 1$, as

$$c = (m/2)^{1/3} g^{1/6} \left\{ \left(\frac{2}{mg^2}\right)^{1/3} E \left[\mathrm{Ai}\left(-\left(\frac{m}{mg^2}\right)^{1/3} E\right)\right]^2 + \left[\mathrm{Ai}'\left(-\left(\frac{2}{mg^2}\right)^{1/3} E\right)\right]^2 \right\}^{-1/2}. \tag{4}$$

Solutions are written based on the parity of the principal quantum number, n,

$$A_e(x) = c\,\mathrm{Ai}\left[(2m^2g)^{1/3}|x| - \left(\frac{2}{mg^2}\right)^{1/3} E_e\right], \tag{5}$$

for even parity, and

$$A_o(x) = c\,\mathrm{sgn}(x)\,\mathrm{Ai}\left[(2m^2g)^{1/3}|x| - \left(\frac{2}{mg^2}\right)^{1/3} E_o\right], \tag{6}$$

for odd parity, with $\mathrm{sgn}(x)$ the signum function. [13]

Note the appearance of the energy in the expressions for the wave function and the normalization factor given above. The energy spectrum of the particle must be determined in order to write out the wave function explicitly, which will be addressed in the section following.

Flügge has solved two similar problems [10] which are instructive to compare to our problem, one-dimensional motion of a particle in a uniform gravitational field over a reflective boundary and of a particle in an accelerating electric field. The particle in a gravitational field over a reflective boundary exhibits only odd-parity solutions (and corresponding energy eigenvalues) that match the odd-parity solutions that we have obtained. In the same spirit, our solutions may be viewed as one-dimensional electron motion in the presence of a grid electode and have both even and odd parity because the region $x < 0$ is accessible for our quantum particle. Our solution has relevance to work in quantum gravity as the first approximation to a one-dimensional wave function subject to self-gravity.

B. The Energy Spectrum

We must require that our solutions of the Schrödinger Equation have well-behaved first and second derivatives and this requirement results in a discrete energy spectrum with energy eigenvalues determined by the continuity conditions for even and odd eigenfunctions. Beginning with the even parity solution, Equation (5), we can directly evaluate the derivative

$$\frac{dA_e}{dx} = c\,\mathrm{sgn}(x)\,(2m^2g)^{1/3}\,\mathrm{Ai}'\left[(2m^2g)^{1/3}|x| - \left(\frac{2}{mg^2}\right)^{1/3} E_e\right] \tag{7}$$

To ensure continuity of the derivative at the origin, we must have that

$$\mathrm{Ai}'\left[-\left(\frac{2}{mg^2}\right)^{1/3} E_e\right] = 0, \tag{8}$$

that is, the even parity energy eigenvalues are generated by zeros of the derivative of the Airy function. Similarly, for the odd parity solution, Equation (6), continuity of the odd parity wave function at the origin requires that

$$\mathrm{Ai}\left[-\left(\frac{2}{mg^2}\right)^{1/3} E_o\right] = 0 \tag{9}$$

and so the odd parity energy eigenvalues are generated by the zeros of the Airy function. We note in passing, that the Airy and Airy derivative terms in the expression for the normalization constant, Equation (4) alternatively vanish for the odd and even parity eigenfunctions, leading to a useful simplification.

The energy spectrum may also be calculated from the expression,

$$E_{e,o} = \int \psi^* H \psi\, dx = c^2 \int_{-\infty}^{\infty} A_{e,o}(x)\left[-\frac{1}{2m}\frac{d^2}{dx^2} + mg|x|\right] A_{e,o}(x)\, dx, \tag{10}$$

with H the Hamiltonian, and $A_{e,o}(x)$ given by Equations (5-6) and (4). This integral may be evaluated to obtain a surprisingly simple transcendental equation for the energy,

$$\mathrm{Ai}(-b)\,\mathrm{Ai}'(-b) = 0 \tag{11}$$

with

$$b = \left(\frac{2}{mg^2}\right)^{1/3} E. \tag{12}$$

The derivation of this result is given in the appendix. This transcendental equation also reflects the parity behavior of the energy eigenspectrum because solutions of this equation

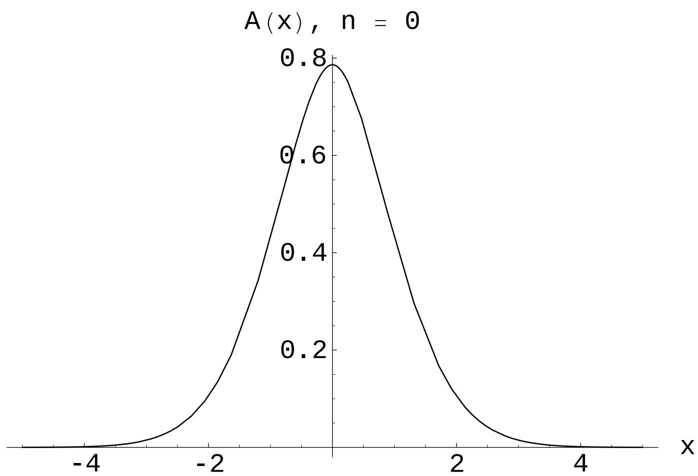

FIG. 1: $A(x)$ for the first energy value corresponding to $n = 0$ and $b = 1.01879$. This is the lowest energy (ground state) solution.

are b values that are either the roots of the Airy function or its derivative. Because of the oscillatory character of the Airy function for negative argument, the solutions of Equation (11) are alternately due to zeros of the Airy function and its derivative. We identify this spectrum of energy eigenvalues with the principal quantum number, n. The first twelve roots of Equation (11) are given in Table I, which yield the energy values in units of $\left(\frac{mg^2}{2}\right)^{1/3}$.

C. Wave Functions for Specific Energy Eigenvalues

We now graph the normalized wave function for the first four energy eigenvalues, which are given in Figures 1 through 4. We take for these plots, $m = g = 1$.

As expected, the $n = 0$ eigenfunction is concentrated at the origin and has a single maximum at $x = 0$. The eigenfunction falls off rapidly as $|x|$ increases, going like $e^{-(2\sqrt{2}/3)|x|^{3/2}}$ as $x \to \pm\infty$ due to the Airy function character of the wave function. Successively higher n eigenfunctions incorporate more of the oscillatory behavior of $\mathrm{Ai}(x)$ for negative x because of the larger values of b for larger n values. Thus, there are more oscillations near the origin for the wave function, before it begins its rapid decay controlled by the large x behavior of $\mathrm{Ai}(x)$. Solutions with even values of n have even parity and those with odd values of n have odd parity; a behavior exhibited by the simple harmonic oscillator solution (see further comparisons in the section following).

Lastly, we show the $n = 11$ eigenfunction in Figure 5. This shows the trend towards more

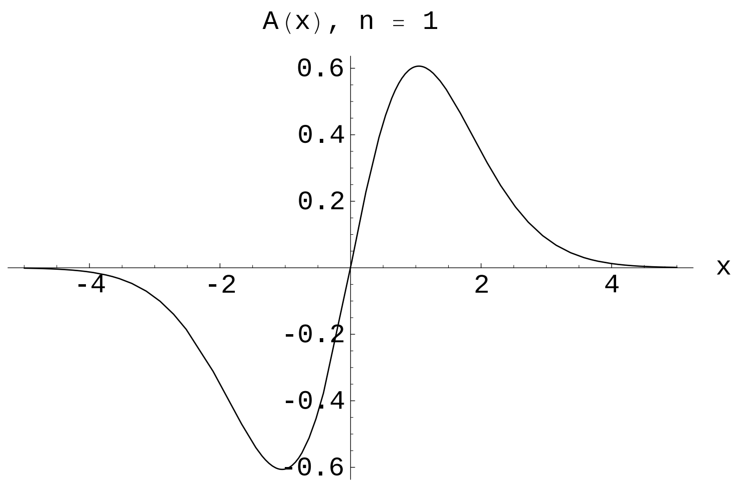

FIG. 2: $A(x)$ for $n = 1$ and $b = 2.33811$. This is the lowest energy odd parity wave function.

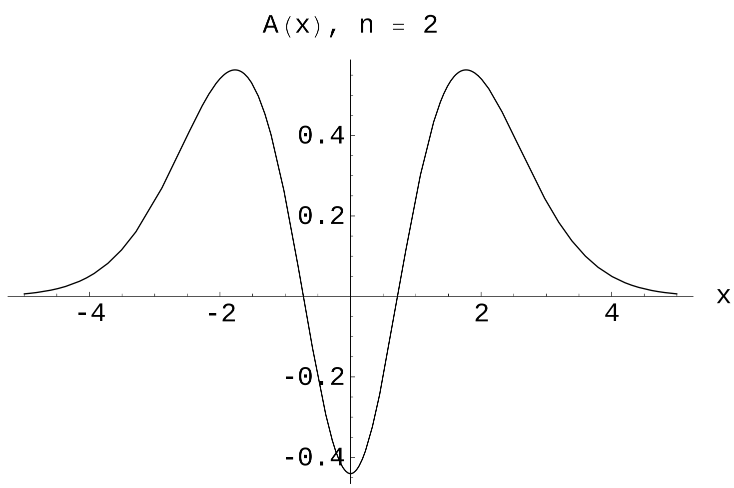

FIG. 3: $A(x)$ for the third energy value corresponding to $n = 2$ and $b = 3.2482$.

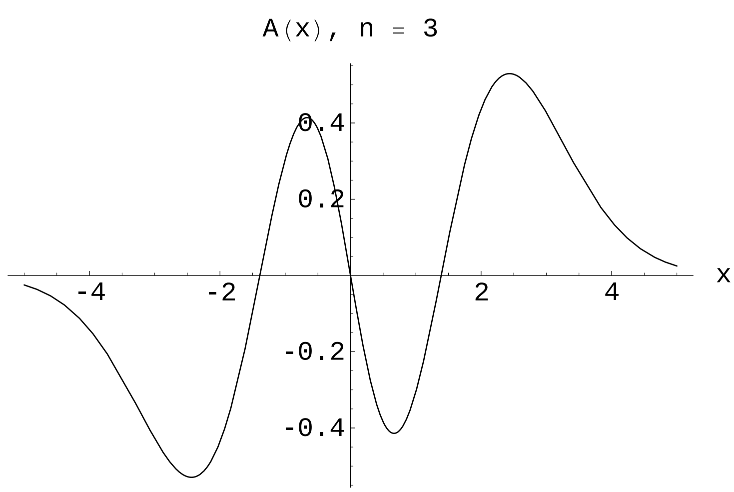

FIG. 4: $A(x)$ for the fourth energy value corresponding to $n = 3$ and $b = 4.08795$.

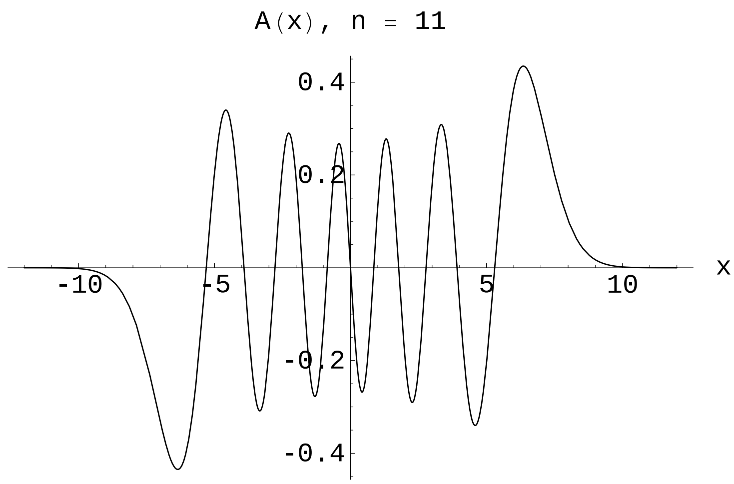

FIG. 5: $A(x)$ for the twelfth energy value corresponding to $n = 11$ and $b = 9.02265$. This plot shows the progressively number of oscillations in the wave function as the principal quantum number increases.

oscillations as n increases. The eigenfunction plots shown here have a strong resemblance to the eigenfunctions for the well-known harmonic oscillator solution of the Schrödinger Equation, plotted by Schiff [11] and others.

Finally, we can use these plots of the energy eigenfunctions of the $g|x|$ potential to verify that the signum function in Equation (6) does not introduce singular behavior at the origin when substituted into the Schrödinger Equation. Note that if we have $g(x) = f(x)\,\mathrm{sgn}(x)$, then $g'(x) = f'(x)\,\mathrm{sgn}(x) + 2f(0)\delta(x)$. For our odd parity solutions the second term on the right hand side vanishes, and thus no singular behavior is introduced at the origin. In the even parity solutions, the signum function is raised to an even power, and so is unity identically and does not impact the solution anywhere. [14]

III. COMPARISON TO THE SIMPLE HARMONIC OSCILLATOR

A. Qualitative Comparison of Eigenfunctions

We expect the rough appearance of these eigenfunctions to correspond to the eigenfunctions of the harmonic oscillator problem, at least for the lower quantum number states. In effect, in making this qualitative comparison, we are considering the $mg|x|$ potential as a "perturbation" or "distortion" of the harmonic oscillator potential, and looking at the corresponding eigenfunctions. Considering the qualitative characteristics of the harmonic oscillator eigenfunctions as plotted in [11] and [12] and many other places, it is apparent that the general properties of solutions having alternating parity and having n nodes with increasing principal quantum number n is common to the harmonic oscillator eigenfunctions and to our new solutions. The parity behavior is perhaps the most important common property because our solution requires eigenfunctions with either zero derivative or zero value at the origin, so as to avoid singular behavior arising from the $|x|$ in the argument of the Airy function.

The harmonic oscillator eigenfunctions have the general character of a Hermite polynomial multiplied times a Gaussian function. Therefore, the harmonic oscillator eigenfunctions fall off more rapidly as $x \to \pm\infty$ than our new solutions which fall off like $e^{-(2\sqrt{2}/3)|x|^{3/2}}$. The more rapid fall off of the harmonic oscillator eigenfunctions is expected because of the more rapid growth of the harmonic oscillator potential $\propto x^2$, compared to the $|x|$ dependence of our potential.

B. Energy Eigenvalues and Large n Behavior

It is well known that the energy eigenvalues of the harmonic oscillator are given by

$$E_n = (n + \frac{1}{2})\hbar\omega_c, \tag{13}$$

with ω_c the angular frequency of the corresponding classical oscillator. Thus the energy eigenvalues of the harmonic oscillator remain evenly spaced for any value of the principal quantum number, n. Examination of Table 1 shows that the first few eigenvalues in our problem rise slightly faster than n. The energy eigenvalues are determined alternately by the zeros of $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$, and so the large n spacing of energy eigenvalues is controlled by the asymptotic behavior of the Airy function as $x \to -\infty$. (Recall that the Airy function has zeros only for $x < 0$.) The leading asymptotic behavior of the Airy function as $x \to -\infty$ is

$$\sin\left[\frac{\pi}{4} + \frac{2}{3}(-x)^{3/2}\right]. \tag{14}$$

This implies that in our problem the energy eigenvalues crowd together like $E_n \sim n^{2/3}$. This is reasonable behavior in view of the fact that at large $|x|$, the x^2 potential grows faster than the $|x|$ potential. The sublinear growth of the energy eigenvalues as n increases is shown

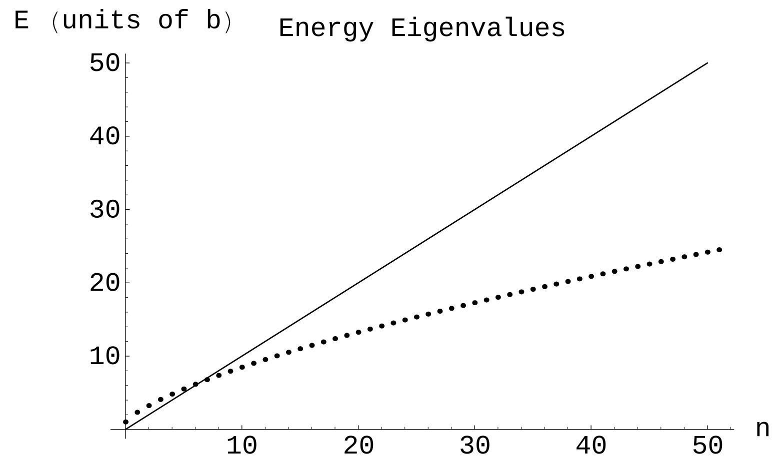

FIG. 6: Energy eigenvalues for $n = 0$ to $n = 51$. The line $E = n$ is plotted for comparison. While the first few energy eigenvalues increase at a rate faster than $E = n$, the rate rapidly slows to $E_n \sim n^{2/3}$, reflecting the crowding of energy eigenvalues at $n \to \infty$. The energy eigenvalues of the quantum harmonic oscillator would all lie $\frac{1}{2}$ unit above the reference line.

in the plot in Figure 5. Flügge calculated a similar crowding of energy eigenvalues in his solution of one-dimensional motion in a uniform gravitational field. [10]

Finally, we examine the behavior of our solution in the correspondence limit as $n \to \infty$. The probability density function of a large n eigenfunction must approach, in a suitable mean sense, the probability density function of a particle following a classical trajectory. The classical probability density function, $f_{\text{class}}(x)\,dx$ measures the probability of finding the particle between x and $x + dx$, averaging over all possible phases of the periodic motion of the particle (or alternately, considering only integral numbers of the half-period of the periodic motion). The classical turning points are at $\pm E/mg$ and the classical period is $4\sqrt{2E/mg}$. In order to make the correct correspondence limit, E must be taken to be one of the energy eigenvalues of the quantum problem, for suitably large n. In Figure 7 we plot the classical and quantum probability density functions for $n = 10$ and $n = 20$. The classical turning points for each problem are indicated by vertical dashed lines. This plot is similar to Figure 11 in Schiff [11] for the harmonic oscillator, however, where Schiff plots the probabilities on an arbitrary scale, we have taken care to plot both the quantum and classical probability density functions on the same normalized scales. The classical probability density function tracks through the mean of the oscillations in the quantum probability density function, which goes to its largest values near the classical turning points, which some quantum "leakage" beyond those points. As n becomes larger, the passage to this qualitative "continuum" limit becomes better, as observed for the harmonic oscillator as well.

The first WKB approximation gives the correct asymptotic behavior of the wave functions as discussed in [10]. This is not surprising because the Airy function provides the matching of the wave function at turning points in the standard treatment of quantum mechanics in the WKB approximation.

Following Pauling and Wilson's reasoning [12] the classical microcanonical distribution approximates the quantum probability density in a manner similar to that of the simple harmonic oscillator, but somewhat more smoothly due to the crowding of the energy levels in the $|x|$ potential. Thus, one expects a more rapid approach to a classical limit in our particular problem.

The fact that the force is discontinuous in our problem does not give rise to difficulties in establishing the Heisenberg picture or in developing the Ehrenfest theorem.

IV. CONCLUSIONS

We have identified an exact, closed form analytic solution of the Schrödinger equation with a potential term of the form $mg|x|$. The solution for the eigenfunctions is in terms of the Airy function in the form $[\mathrm{sgn}(x)]^n \,\mathrm{Ai}(a|x| - b)$, with a and b constants and n the principal quantum number. Values of b are determined by the energy eigenvalue problem. The eigenfunctions obtained in this problem are a sequence of alternating parity functions reminiscent of the well-known quantum harmonic oscillator. Values of the energy eigenvalues are controlled alternately by zeros of the Airy function and its derivative. Due to the properties of the Airy function, the energy eigenvalues become more closely spaced as $n \to \infty$, unlike the harmonic oscillator eigenvalues, which remain evenly spaced. We believe that this exact solution of Schrödinger's equation will have utility in future work in problems in solid-state physics and in quantum gravity.

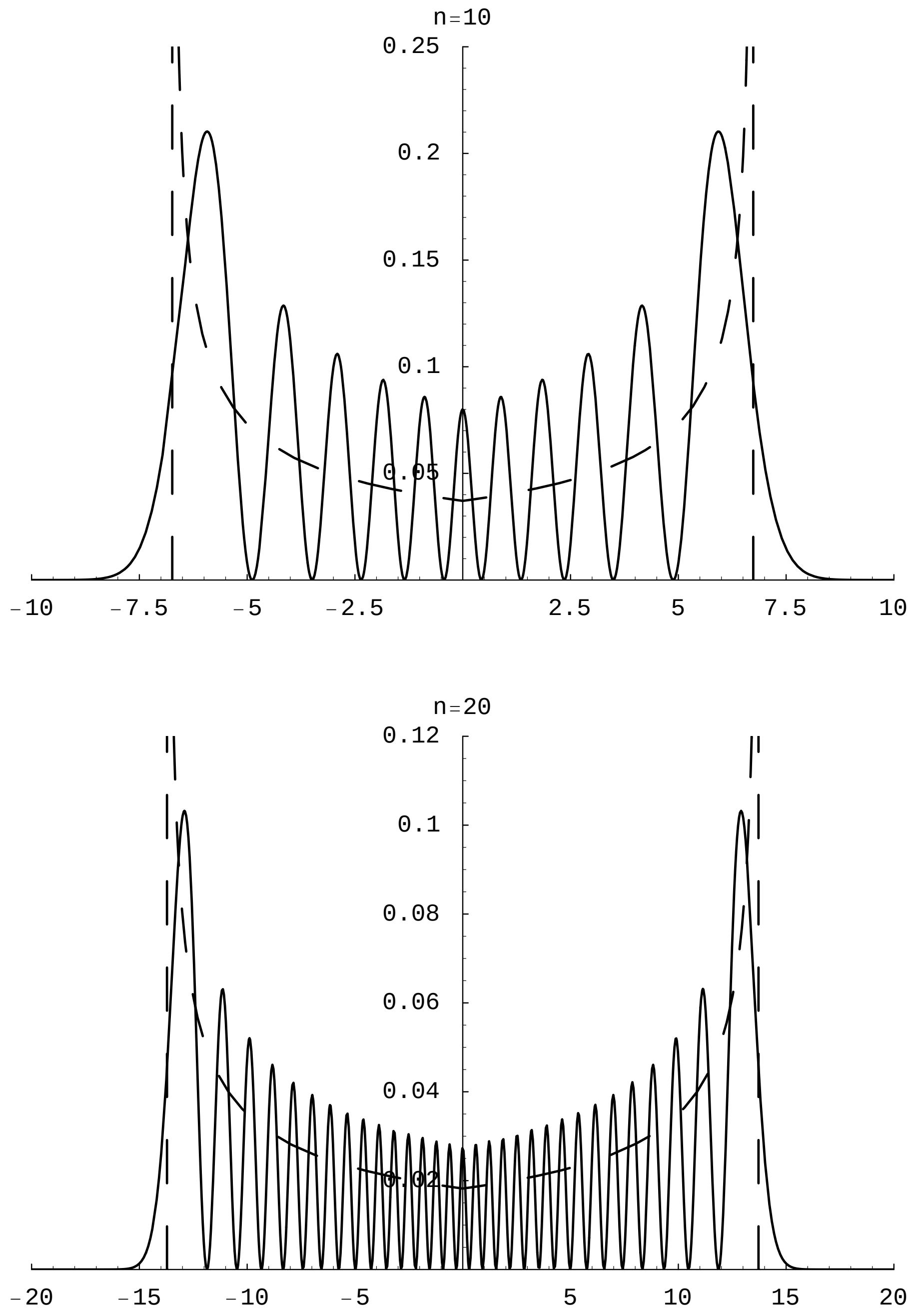

FIG. 7: Quantum probability density distributions (solid line) and corresponding classical probability density distributions (dashed line) for the exact solution for the $|x|$ potential. The upper pane shows the quantum solution for $n = 10$ together with the classical probability density distribution for the same energy. The lower pane shows the corrsponding plots for $n = 20$. The correspondence limit behavior of this system, similar to that of the harmonic oscillator solution, may be seen.

Acknowledgments

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APPENDIX: OPERATOR CALCULATION OF THE ENERGY SPECTRUM OF THE $g|x|$ POTENTIAL

For concreteness, we perform the calculation for an even-parity wave function and indicate how the odd-parity case appears in the same framework. Write the energy equation with the Hamiltonian given explicitly given (and recall that we can take the wave function to be real without loss of generality),

$$\begin{aligned} E &= \int_{-\infty}^{\infty} A(x) H A(x)\, dx \\ &= \int_{-\infty}^{\infty} A(x) \left[-\frac{1}{2m}\frac{d^2}{dx^2} + mg|x| \right] A(x)\, dx \\ &= c^2 \int_{-\infty}^{\infty} \mathrm{Ai}(a|x|-b) \left[-\frac{1}{2m}\frac{d^2}{dx^2} + mg|x| \right] \mathrm{Ai}(a|x|-b)\, dx \end{aligned} \tag{A.1}$$

with c the normalization constant, $a = (2m^2g)^{1/3}$, and $b = (2/mg^2)^{1/3}$. This gives us two integral terms which we treat in turn.

$$E = -\frac{c^2}{2m} \int_{-\infty}^{\infty} \mathrm{Ai}(a|x|-b) \frac{d^2}{dx^2} \mathrm{Ai}(a|x|-b)\, dx + c^2 g \int_{-\infty}^{\infty} |x| \left[\mathrm{Ai}(a|x|-b)\right]^2\, dx \tag{A.2}$$

We begin with the first integral in Equation (A.2). Calculating the second derivative of the Airy function,

$$\frac{d^2}{dx^2}\mathrm{Ai}(a|x|-b) = 2a\mathrm{Ai}'(-b)\delta(x) + a^2 Ai''(a|x|-b), \tag{A.3}$$

where $\delta(x)$ is Dirac's delta function. Observing from Airy's Equation that

$$\frac{d}{dz}\mathrm{Ai}'(z) = z\mathrm{Ai}(z), \tag{A.4}$$

we obtain,

$$\frac{d^2}{dx^2}\mathrm{Ai}(a|x|-b) = 2a\mathrm{Ai}'(-b)\,\delta(x) + a^2(a|x|-b)\mathrm{Ai}(a|x|-b). \tag{A.5}$$

Substituting into the equation for E and evaluating,

$$E = -\frac{c^2}{m} a \operatorname{Ai}(-b) \operatorname{Ai}'(-b) - \frac{c^2 a}{2m} \int_{-\infty}^{\infty} (a|x| - b) \left[\operatorname{Ai}(a|x| - b)\right]^2 dx + c^2 g \int_{-\infty}^{\infty} |x| \left[\operatorname{Ai}(a|x| - b)\right]^2 dx. \tag{A.6}$$

The second and third terms of this equation are evaluated using the integrals,

$$\int_{-\infty}^{\infty} (a|x| - b) \left[\operatorname{Ai}(a|x| - b)\right]^2 dx = -\frac{2}{3a} \left\{ b^2 \left[\operatorname{Ai}(-b)\right]^2 + \operatorname{Ai}(-b) \operatorname{Ai}'(-b) + b \left[\operatorname{Ai}'(-b)\right]^2 \right\} \tag{A.7}$$

and

$$\int_{-\infty}^{\infty} |x| \left[\operatorname{Ai}(a|x| - b)\right]^2 dx = \frac{1}{3a^2} \left\{ 4b^2 \left[\operatorname{Ai}(-b)\right]^2 - 2\operatorname{Ai}(-b) \operatorname{Ai}'(-b) + 4b \left[\operatorname{Ai}'(-b)\right]^2 \right\}. \tag{A.8}$$

Substituting into Equation (A.6), we have

$$E = -\frac{c^2}{m} a \operatorname{Ai}(-b) \operatorname{Ai}'(-b) + \frac{c^2}{3m} \left\{ b^2 \left[\operatorname{Ai}(-b)\right]^2 + \operatorname{Ai}(-b) \operatorname{Ai}'(-b) + \left[\operatorname{Ai}'(-b)\right]^2 \right\} + \frac{2c^2 g}{3a^2} \left\{ 2b^2 \left[\operatorname{Ai}(-b)\right]^2 - \operatorname{Ai}(-b) \operatorname{Ai}'(-b) + 2b \left[\operatorname{Ai}'(-b)\right]^2 \right\}. \tag{A.9}$$

This expression simplifies to

$$E = c^2 \frac{a}{m} \left\{ b^2 \left[\operatorname{Ai}(-b)\right]^2 - \operatorname{Ai}(-b) \operatorname{Ai}'(-b) + b \left[\operatorname{Ai}'(-b)\right]^2 \right\}. \tag{A.10}$$

From Equation (4), we can write c^2 in terms of $\operatorname{Ai}(-b)$ and $\operatorname{Ai}'(-b)$ to obtain,

$$b = \frac{b^2 \left[\operatorname{Ai}(-b)\right]^2 - \operatorname{Ai}(-b) \operatorname{Ai}'(-b) + b \left[\operatorname{Ai}'(-b)\right]^2}{b \left[\operatorname{Ai}(-b)\right]^2 + \left[\operatorname{Ai}'(-b)\right]^2}, \tag{A.11}$$

and Equation (11) is immediate.

The calculation for odd parity states is analogous, but simpler because the requirement that the wave function be continuous at the origin eliminates singular terms.

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[13] Prof. Barton suggests that the odd-even parity structure of this solution might be expected from Sturm-Liouville theory because of the potential being an even function.

[14] This guarantees our solution is a C^2 function.

TABLE I: Energy E measured in units of $\left(\frac{mg^2}{2}\right)^{1/3}$ for principal quantum number $n = 0$ to $n = 11$.

n	E
0	1.01879
1	2.33811
2	3.2482
3	4.08795
4	4.8201
5	5.52056
6	6.16331
7	6.78671
8	7.37218
9	7.94413
10	8.48849
11	9.02265